QUADRATIC FORMS WHICH REPRESENT ALL INTEGERS

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We shall give generalizations of the classic theorem that every positive integer is a sum of four squares. We seek all sets of positive integers $a, b, \ldots$ such that every positive integer can be expressed in the form $f = ax^2 + by^2 + \ldots$. We may arrange the terms so that $a \leqq b \leqq c \ldots$ Since f shall represent 1 and 2, evidently $a = 1$, $b < 3$. Hence either $f = x^2 + y^2 + cz^2 + \ldots$ or $f = x^2 + 2y^2 + cz^2 + \ldots$. If $c > 3$ in the former, then $f \neq 3$. If $c > 5$ in the latter, then $f \neq 5$. Hence the sum of the first three terms of f is one of the following ternary forms:

$$t_1 = x^2 + y^2 + z^2, \quad t_2 = x^2 + y^2 + 2z^2, \quad t_3 = x^2 + y^2 + 3z^2,$$
$$t_4 = x^2 + 2y^2 + 2z^2, \quad t_5 = x^2 + 2y^2 + 3z^2, \quad t_6 = x^2 + 2y^2 + 4z^2$$
$$t_7 = x^2 + 2y^2 + 5z^2.$$

The least positive integer l_i not represented by t_i is as follows: $l_1 = 7$, $l_2 = 14$, $l_3 = 6$, $l_4 = 7$, $l_5 = 10$, $l_6 = 14$, $l_7 = 10$. This proves that not all positive integers are represented by any form $ax^2 + by^2 + cz^2$ in which a, b, c are positive integers. It proves also that, if $f = t_i + d_iu^2 + \ldots$ represents all positive integers, then $d_i \leqq l_i$. By hypothesis, d_i is not less than the coefficient of z^2. Hence the first four coefficients of f are those in one of the following 54 sets:

$$\begin{array}{ll} 1,1,1,s\ (s = 1, \ldots, 7); & 1,2,2,s\ (s = 2, \ldots, 7); \\ 1,1,2,s\ (s = 2, \ldots, 14); & 1,2,3,s\ (s = 3, \ldots, 10); \\ 1,1,3,s\ (s = 3,4,5,6); & 1,2,4,s\ (s = 4, \ldots, 14); \\ & 1,2,5,s\ (s = 6, \ldots, 10), \end{array} \qquad (1)$$

or else 1,2,5,5. But the latter are the coefficients of

$$q = x^2 + 2y^2 + 5z^2 + 5u^2,$$

which does not represent 15, since $x^2 + 2y^2$ does not represent 5, 10 or 15, while $z^2 + u^2 \neq 3$. This exceptional value was overlooked by Ramanujan.[1] Without proof he stated empirical theorems on the forms of numbers represented by $t_1, \ldots, t_7$. Recently these theorems have been completely proved by the writer.[2] From them we readily conclude that every positive integer can be represented by each of the 54 forms in four variables whose coefficients are given by (1), and that q represents all positive integers except 15.

Consider a form $f = ax^2 + by^2 + \ldots$ in $n \geqq 5$ variables which represents all positive integers. We have shown that the first four coefficients are 1,2,5,5 or a set (1). In the latter case the remaining coefficients are arbitrary. The only interesting case is, therefore, $f = q + ev^2 + \ldots$ such that no quaternary form, obtained by deleting all but four terms of f, represents all integers. Then $e < 16$, since otherwise $f = 15$ would require $q = 15$, which was seen to be impossible. The values 6,7,8,9,10 of e are excluded, since the abridged form $x^2 + 2y^2 + 5z^2 + ev^2$ was seen to represent all positive integers. Hence $e = 5,11,12,13,14$ or 15. In these respective cases, we see that f represents 15 when $z = u = v = 1$; $x = 2, v = 1$; $x = y = v = 1$; $y = v = 1$; $x = v = 1$; $v = 1$; with all further variables zero. We may, therefore, state the

THEOREM. *If, for* $n \geqq 5$, $f = a_1x_1^2 + \ldots + a_nx_n^2$ *represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then* $n = 5$ *and* $f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2$ $(e = 5,11,12,13,14,15)$, *and these six forms f actually have the property stated.*

[1] Ramanujan, *Proc. Cambridge Phil. Soc.*, **19**, 11 (1916–9).

[2] Dickson, *Bull. Amer. Math. Soc.*, **33** (1927).

CONGRUENCES OF PARALLELISM OF A FIELD OF VECTORS

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1. In a geometry of paths the equations of the paths may be written in the form

$$\frac{dx^i}{dt}\left(\frac{d^2x^j}{dt^2} + \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) - \frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) = 0, \quad (1.1)$$

t being a general parameter and Γ^i_{kl} functions of the k's which are symmetric in k and l; a repeated Latin index indicates the same from 1 to n of that index. These functions serve to define infinitesimal parallelism of vectors and accordingly are called the coefficients of the affine connection. We may go further and say that if λ^i are the components of a field of contravariant vectors and C is any curve of the space (at points of which the x's are given as functions of a parameter t), the vectors of the field at points of C are parallel with respect to C, when, and only when,

$$(\lambda^h\lambda^i_{,j} - \lambda^i\lambda^h_{,j})\frac{dx^j}{dt} = 0, \quad (1.2)$$

where

$$\lambda^i_{,j} = \frac{\partial\lambda^i}{\partial x^j} + \lambda^h\Gamma^i_{hj}. \quad (1.3)$$